EFFECT OF CHANGE IN INNER PRODUCT ON SPECTRAL ANALYSIS OF CERTAIN LINEAR OPERATORS IN HILBERT SPACES ARISING FROM ORTHOGONAL POLYNOMIALS

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Abstract. In this paper we present how spectral properties of certain linear operators vary when operators are considered in different Hilbert spaces having common dense domain as the space of polynomials in one real variable with complex coefficients. This is done taking differential operator and matrix operator representation of dilation operator $E_{p,d}$ dilating p, a polynomial sequence, by d, a non-constant sequence of non-zero complex scalars. For the purpose of identification of $E_{p,d}$ with formal differential operator, which includes finite order differential operators as special cases, we derive conditions under which a formal differential operator has a polynomial sequence as a sequence of eigen functions corresponding to non-zero eigen values. As a consequence we get, for instance,

$$((1/72)x^4 - 3x)y^{(4)} - xy^{(2)} - (1/3)xy^{(1)} + (4/3)y.$$

does not have any polynomial sequence as such a sequence of eigen functions, since no fourth order polynomial is such an eigen svector.
Keywords: Dilation Operator, Formal Linear Differential Operator, Matrix Operator, Ordinary Differential Operator, Shift Operator, Orthogonal Polynomials, Closure of an Operator, Spectrum, Eigen Functions.
2010 AMS Subject classification: 33C45, 34L05, 34L10, 47A05, 47A10, 47A58, 47A75

1. Introduction

Let $\mathcal{P}_c$ be the linear space of all polynomials in one real variable with complex co-efficients ; $\mathcal{P}$ be the set of sequences $p = (p_n)_{n\in\mathbb{N}_0}$, $p_n \in \mathcal{P}_c$, $\forall\ n \in \mathbb{N}_0$, $p_0 \equiv 1$ and $p_n(x) = \sum_{k=0}^{n} p_{nk}x^k$, $p_{nn} \neq 0$, $n \geq 1$; $\mathbb{N}_0$ is the set of non-negative integers. Let $\mathcal{P}_r$ be the real linear space consisting of real polynomials in one real variable. We take $\mathcal{P}_o = \{p \in \mathcal{P} :$ p is an orthogonal polynomial sequence$\}$. Any $p \in \mathcal{P}$ is called a PS and OPS stands for orthogonal polynomial sequence ; orthogonality is always with respect to a positive definite moment functional as in [1]. We denote by $\widetilde{D}$ the set of non-constant sequences $d = (d_n)_{n\in\mathbb{N}_0}$ of non-zero complex scalars. Set $D = \{d \in \widetilde{D} :$ d is a real sequence with $d_0 = 1\}$.

In this paper we consider dilation operators, in particular $S_{p,d}$ as defined and studied in [3]. Definition of $S_{p,d}$ was motivated by similar operators in literature , for instance, in [5] and [6]. For an OPS, $p \in \mathcal{P}_o$ and $d \in D$, $S_{p,d}$ is the linear map on $\mathcal{P}_c$ to itself such that $S_{p,d}(p_n) = d_n p_n,\ n \in \mathbb{N}_0$. Dilation map $E_{p,d}$ is like $S_{p,d}$ with p as a PS and $d \in \widetilde{D}$. For more facts on $S_{p,d}$ and $E_{p,d}$, we refer the reader to [3], [7] and to Appendix C.
For convenience of readers, relevant concepts and results needed throughout the paper have been freely collected from standard sources such as [1], [4], [8], [9] and research papers such as [3], [5], [6] and [7] and then, put as Appendices A to D given at the end of the paper.
We begin with giving some results from [5] and [6] in section 1. These motivated us to study and consider different representations of dilation operators.
In sections 2 and 3 we discuss and derive conditions when a formal differential operator, denoted as η in this paper, has a dilation operator representation. As is well-known, for instance refer to [5] and Appendix B.1(i), $\eta : \mathcal{P}_c \longrightarrow \mathcal{P}_c$ is the operator defined by the formal sum

$$\eta(y)(x) = \sum_{k=0}^{\infty} M_k(x) y^{(k)}(x)$$

where $M_k(x) \in \mathcal{P}_c$ is such that $\deg M_k(x) \leq k$ for $k \in \mathbb{N}$, $M_0(x)$ is a constant, $y^{(0)} = y(x)$, $y^{(k)}(x)$ is the k^{th} derivative of $y(x) \in \mathcal{P}_c$ for $k \geq 1$. Let $m : \mathcal{P}_c \longrightarrow \mathcal{P}_c$ be the operator η for which $M_k(x) \equiv 0,\ \forall\ k > r$ for some smallest number $r \in \mathbb{N}$ with $M_r \not\equiv 0$. The number r is then referred to as the order of finite order differential operator m. $\widehat{\eta}$ is an operator η having a PS as a sequence of eigen functions corresponding to a sequence of non-zero eigenvalues. Similar meaning is attached to $\widehat{m}$.
In section 4 differential operator representation of dilation operator $S_{p,d}$ enables us to obtain a result about shift operator $\tau_{a,b}, a, b \in \mathbb{R}, a \neq 0$, defined as $\tau_{a,b}(x^n) = (ax+b)^n, n \in \mathbb{N}_0$, extended linearly to $\mathcal{P}_c$. It turns out that if such an operator has a sequence of orthogonal polynomials as a sequence of eigen functions then these polynomials are necessarily translates of some symmetric polynomial sequence with the corresponding sequence of eigen values just as $((-1)^n)$.
In section 5 dilation operator representation of differential operator enables us to do spectral analysis, in certain cases, by applying the Thin infinite matrix theory developed by the author and Ajit Iqbal Singh in [7]. Appendix D contains relevant definitions and results from [7]. In this section the setting changes from $\mathcal{P}_c$ to the Hilbert Space $H(q)$ arising from a PS q in the following manner. Starting with a PS $q = (q_n)$, we identify a complex sequence $a = (a_n)_{n \in \mathbb{N}_0}$, having finitely many non-zero terms, with $\sum_{n \in \mathbb{N}_0} a_n q_n$. Then completion of $\mathcal{P}_c$ with the inner product induced by the usual inner product in l_2 is a Hilbert Space which we denote by $H(q)$. If q is taken as an OPS, then $\tilde{q}$ is the corresponding orthonormal sequence.

We may note these spaces include Lebesgue Spaces $\mathcal{L}_\omega^{(2)}(a,b)$ whenever q is an OPS with respect to weight function ω [1]. We begin with the question of closability of operators considered in this section. We then look at the spectral properties of operators which are closable and observe they display varied situations regarding spectra, closures and adjoints. In one instance $E_{p,d}$ in $H(q)$ has the whole of complex plane $\mathbb{C}$ in its spectrum, major part of which constitutes the continuous spectrum. There are instances where the domain of adjoint may be dense, yet no basic vector may be in it. Operators considered include finite order differential operators associated with classical orthogonal polynomials such as the ones given in Appendix B.3. We also observe that availability of closures, in some instances, is heavily dependent on eigen sequence d and in some others nature of d does not matter much.

2. Motivation and Discussion

We begin with quoting two results, Proposition MSZ from [6], Lemma KS from [5], in both retaining the original notation as far as possible.

Proposition MSZ ([6]). *For a degree preserving linear operator S, the following statements are equivalent.*

(i) $s = S(1)$ *and* $S(y) = sy \circ \tau_{a,b}$, y *belonging to* $\mathcal{P}_r$, $s \neq 0$, $a \neq 0$, $b \in \mathbb{R}$.

(ii) *If* $(p_n)_{n=0}^{\infty}$ *satisfies the three-term recurrence relation*

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x),$$

then $(Sp_n)_{n=0}^{\infty}$ *satisfies the three-term recurrence relation with co-efficients* $\alpha_n = a^{-1}a_n$, $\beta_n = a^{-1}(b_n - b)$ *and* $\gamma_n = a^{-1}c_n$, *respectively.*

Following lemma from [5] strongly indicates differential operator representation of the dilation operator.

Lemma KS ([5]). *Consider the equation*

$$(L(y))(x) = \sum_{k=1}^{\infty} M_k(x) y^{(k)}(x) = \lambda y(x), \tag{1}$$

where each $M_n(x)$ *is a polynomial of degree* $\leq n$

$$M_n(x) = m_{n,0} + m_{n,1}x + \ldots + m_{n,n}x^n\,. \tag{2}$$

Let

$$\lambda_n = nm_{1,1} + n(n-1)m_{2,2} + \ldots + n!m_{n,n} \tag{3}$$

and assume that

$$\lambda_m \neq \lambda_n \quad (m \neq n). \tag{4}$$

Then

(i) *For* $\lambda \neq \lambda_n$ $(n = 0, 1, \ldots)$ *the only polynomial satisfying* (1) *is* $y \equiv 0$. *For* $\lambda = \lambda_n$ *there is a polynomial solution of* (1)*, unique to within an arbitrary constant multiplier, and it is of degree* n.

(ii) *Conversely, let* $(P_n(x))$ *be an arbitrary PS and let* (λ_n) *be an arbitrary sequence of constants with* $\lambda_n \neq 0$, $\lambda_m \neq \lambda_n$ $(m \neq n)$. *Then there exists a unique sequence* $(M_n(x))$ *of polynomials with degree* $M_n \leq n$, *such that* (1) *has* $(y_n = P_n(x))$ *as its PS of solutions.*

Remark 1. For the purpose of identification of $E_{p,d}$ with η through the lemma, we first note the following facts:

(i) Though λ_0 is not defined in Lemma KS [5], WLOG we take it as zero throughout. Since $M_0 p_0 = d_0 p_0$, $p_0 \equiv 1$, $\forall$ PS p we get $M_0 = d_0$, η is $L + M_0 I$, consequently in place of λ_n in the lemma we have $d_n - d_0$, $n \in \mathbb{N}$.

(ii) Part (i) then implies, for an $n \geq 1$, $\lambda_n = 0 \Longleftrightarrow d_n = d_0$ and for $m, n \in \mathbb{N}_0$, $m \neq n$, $\lambda_m \neq \lambda_n \Longleftrightarrow d_m \neq d_n$.

(iii) We recall from [7], as in Appendix C.1(i), for a fixed $p \in \mathcal{P}_o$, $d \in D$, $S_{p,d} = S_{q,d}$ may be true for infinite number of q. We shall, therefore, consider the triplet (η, p, d) for differential operator representation of $E_{p,d}$. Also, we treat the question of existence and that of uniqueness separately for any one of (η, p, d) given $p \in \mathcal{P}$, $d \in \widetilde{D}$.

(iv) (a) Given (p, d), unique η may be obtained recursively as

$$M_1 p_1^{(1)} = (d_1 - d_0) p_1,$$

$$M_k p_k^{(k)} = -\sum_{j=1}^{k-1} M_j p_k^{(j)} + (d_k - d_0) p_k, \quad k > 1, \quad k \in \mathbb{N}.$$

Expanded form of the equations are given in Remark 3.

(b) From Appendix B.3.(iii), we see that $m^\alpha(L_n^\alpha) = (-2n + 1) L_n^\alpha$, $n \in \mathbb{N}_0$. Uniqueness of η then implies
$S_{L_n^\alpha, d} \equiv m^\alpha$
on $\mathcal{P}_c$ for $d = (-2n + 1)$.
Same holds for $m^{\alpha,\beta}$ and m^h, given in Appendix B.3(i),(ii).

(v) It follows from Lemma KS that a necessary condition on $d \in \widetilde{D}$ for (η, p, d) to be such that $\eta p = dp$ is that

$$d_n - d_0 = \sum_{r=1}^{n} m_{rr} p(n, r), \quad n \in \mathbb{N}.$$

(vi) Above condition on d will be assumed without stating it when a solution to $\eta p = dp$ is sought. Also, $\sum\limits_{r=1}^{n} m_{rr} p(n, r) \neq -d_0$ is implied.

3. Differential operator representation of (p,d)-dilation $E_{p,d}$

In this section, we discuss the formal differential operator η and obtain conditions under which such an operator has a PS as a sequence of eigen functions corresponding to a sequence of non-zero eigen values. We refer to [4], book by A.M. Krall, instead of original sources ; chapter XVII of the book contains a survey of results related to formal infinite order differential operators and associated classical-type orthogonal polynomials. In Appendix B.2, we give η having Generalized Laguerre type polynomials (defined and studied by T.H.Koornwinder) as a sequence of eigen functions. Following Proposition identifies all η which have a PS as such a sequence of eigen-functions. This, in turn, identifies all differential operators η which do not have a PS as a sequence of eigen functions corresponding to a sequence of non zero eigen values. We also follow it up by a discussion of η and m. To ensure existence of solutions to $\eta p = dp$, it is enough to obtain solutions to $\eta p_n = d_n p_n$ through induction on n, since $p_n^{(k)} = 0$, for $k > n$. WLOG we assume $p_{nn} = 1$ in Proposition 1.

Proposition 1. *Assume $(p_r)_{r=0}^{n-1}$ exist such that $\eta p_r = d_r p_r$. For $k \in \mathbb{N}$, let M_k, $R_{k-1} \in \mathcal{P}_c$ be such that $\deg M_k \leq k$, $\deg R_{k-1} \leq k-1$ and $M_k = m_{kk}x^k + R_{k-1}$. Also, let $\sum\limits_{k=1}^{n} p(n,k) R_{k-1} x^{n-k} = \sum\limits_{j=0}^{n-1} \alpha_j^{n-1} p_j$, where $(\alpha_j^{n-1})_{j=0}^{n-1}$ is an n-tuple of scalars. Then $\exists\ p_n = x^n + q_{n-1}$ for some polynomial q_{n-1} with $\deg q_{n-1} \leq n-1 : \eta p_n = d_n p_n$*

$$\Longleftrightarrow$$

$\exists$ an n-tuple $(\beta_j^{n-1})_{j=0}^{n-1}$ of scalars such that $(d_n - d_j)\beta_j^{n-1} = \alpha_j^{n-1}$, $0 \leq j \leq n-1$.

Proof. $\sum\limits_{k=0}^{n} M_k p_n^{(k)} = d_n p_n$.

$$\Longleftrightarrow \sum_{k=1}^{n} (m_{kk}x^k + R_{k-1}) \left(p(n,k)x^{n-k} + q_{n-1}^{(k)} \right) = (d_n - d_0)(x^n + q_{n-1}).$$

$$\Longleftrightarrow \text{(i) } d_n - d_0 = \sum_{k=1}^{n} p(n,k) m_{kk},$$

$$\text{(ii) } \sum_{k=1}^{n} p(n,k) R_{k-1} x^{n-k} = (d_n - d_0) q_{n-1} - \sum_{k=1}^{n-1} M_k q_{n-1}^{(k)} .$$

$\Longleftrightarrow \exists$ an n-tuple $(\beta_j^{n-1})_{j=0}^{n-1}$ of scalars such that $q_{n-1} = \sum\limits_{j=0}^{n-1} \beta_j^{n-1} p_j$ satisfies

$$\sum_{j=0}^{n-1} \alpha_j^{n-1} p_j = (d_n - d_0) \sum_{j=0}^{n-1} \beta_j^{n-1} p_j - \sum_{j=0}^{n-1} (d_j - d_0) \beta_j^{n-1} p_j .$$

$$\iff (d_n - d_j)\beta_j^{n-1} = \alpha_j^{n-1}, \qquad 0 \le j \le n-1\,. \qquad \square$$

Remark 2. From above, we have

(i) $d_n - d_j = 0 = \alpha_j^{n-1}$, for some $0 \le j < n$ implies non-uniqueness of solutions. In particular, we have the following facts.
  (a) If $d_n = d_0$ for some $n \ge 1$ and p_n is a solution of $\eta p_n = d_n p_n$ then $p_n + c$ is a solution for arbitrary scalar c.
  (b) If $d_t = d_n$ for some $t \ne n$, WLOG, $n > t$, $t, n \ge 1$ and p_n, p_t are such that $\eta p_n = d_n p_n$, $\eta p_t = d_t p_t$, then $\eta(p_n - kp_t) = d_n(p_n - kp_t)$, $k \in \mathbb{C}$.
  (c) If $d_n = (-1)^n, n \in \mathbb{N}_0$, then all symmetric PS are solutions of $\eta p = dp$. In particular, it is so for $p = (P_n^{\alpha,\alpha})$, for any $\alpha > -1$; refer to Appendix A.1.(ii).

(ii) $d_n - d_j = 0$, $\alpha_j^{n-1} \ne 0$, for some $n \in \mathbb{N}_0$ and $j \in \mathbb{N}_0$ with $j < n$, implies non-existence of a required type of solution and this is the only way by which a given η fails to have a PS as a sequence of eigen functions.

Remark 3. (i) Alternative to Remark 2(ii) above, we can obtain all η (in particular m) which do not have a PS as a sequence of eigen-functions from the following recursion equations.

(ii) **We give an expanded form of recursion equations** $\eta p_n = d_n p_n$, $n \in \mathbb{N}_0$.
We first note that for compatibility of equations (a)-(e) below it is essential that

$$(d_n - d_0) \ne -M_0.$$

$$\text{For } n \in \mathbb{N}_0, \text{ we have } \eta p_n = d_n p_n$$

$$\iff$$

(a) $$(d_n - d_0)p_{nn} = m_{nn}p(n,n)p_{nn} + \sum_{r=1}^{(n-1)} m_{rr}p_{nn}p(n,r),$$

(b) $$(d_n - d_0)p_{n\,(n-1)} = m_{n\,(n-1)}p(n,n)\,p_{nn} + \sum_{r=1}^{n-1}[m_{rr}\,p_{n\,(n-1)}p(n-1,r) + m_{r\,(r-1)}p_{nn}p(n,r)],$$

(c) for $2 \le r \le n-2$

$$(d_n - d_0)p_{nr} = m_{nr}\,p_{nn}p(n,n) + \sum_{k=r}^{n-1}\left[\sum_{t=0}^{\min\{n-k,r\}} m_{k\,(r-t)}p_{n\,(k+t)}p(k+t,k)\right] + \sum_{s=1}^{r-1}\left[\sum_{t=0}^{\min\{n-r,s\}} m_{s\,(s-t)}p_{n\,(r+t)}p(r+t,s)\right],$$

(d) $$(d_n - d_0)p_{n1} = m_{n1}p_{nn}p(n,n) + \sum_{r=1}^{(n-1)}[m_{r1}p_{nr}p(r,r) + m_{r0}p_{n\,(r+1)}p(r+1,r+1)],$$

(e) $(d_n - d_0)p_{n0} = m_{n0}\, p_{nn}\, p(n,n) + \sum\limits_{r=1}^{n-1} m_{r0} p_{nr} p(r,r)$.

Remark 4. As a simple application, we get

$$m(y) = \left(12x^4 - 3x\right) y^{(4)} - xy^{(2)} - \frac{1}{3}xy^{(1)} + \frac{4}{3}$$

as one differential operator not having a PS as a sequence of eigen functions. No fourth order polynomial is an eigen function with a corresponding nonzero eigen value.

Discussion 1. Operator $\widehat{m}$, a finite order differential operator has been studied by many authors. In [2], a survey of results concerning classification of $\widehat{m}$ has been done. Finite order and infinite order differential operators are intrinsically very different in many respects. For one, sequence of nonzero eigen-values of a finite order differential operator, when it exists, is an unbounded one. Therefore, for d bounded, $E_{p,d}$ is represented by an infinite order differential operator. The following Proposition points how to construct many such operators.

We point out that Operator in Appendix B.2 is an infinite order differential operator and its sequence of eigen values is an unbounded one.

Proposition 2. *Any finite number of perturbations in the sequence of nonzero eigen-values of a finite order differential operator, with the corresponding sequence of eigen-functions as fixed, makes it necessarily of infinite order.*

Proof. Let (η, p, d) with η as some finite order differential operator be given: $\eta p = dp$. Let $d' \in \widetilde{D}$ be such that for some $n \in \mathbb{N}_0$, $d_k = d'_k$, $0 \le k < n$ and $d_n \neq d'_n$. Also assume that $d_k \neq d'_k$ only for some finite number of indices $k > n$. Then unique $\widetilde{\eta} \approx (\widetilde{M}_k)_{k \in \mathbb{N}_0}$ corresponding to (p, d') is such that for $0 \le j < n$, $\widetilde{m}_{jj} = m_{jj}$, $\widetilde{m}_{n,n} - m_{n,n} = (d'_n - d_n)/n!$ and for $t > 0$,

$$\widetilde{m}_{n+t,n+t} - m_{n+t,n+t} = \frac{d'_{n+t} - d_{n+t}}{(n+t)!} - \sum_{r=n}^{n+t-1} (\widetilde{m}_{r,r} - m_{r,r}) \frac{1}{(n+t-r)!},$$

where $\widetilde{M}_k(x) = \sum\limits_{t=0}^{k} \widetilde{m}_{kt} x^t$, $M_k(x) = \sum\limits_{t=0}^{k} m_{kt} x^t$. Now, if $\widetilde{\eta}$ were also a finite order differential operator, then we would have

$$\widetilde{m}_{n+t,n+t} - m_{n+t,n+t} = 0, \quad \forall\ t > s \quad \text{for some} \quad s \in \mathbb{N}.$$

Also, $d'_{n+t} - d_{n+t} = 0$, $\forall\ t > u$ for some $u \in \mathbb{N}$, by assumption.

Therefore, $\sum\limits_{r=n}^{n+t-1} (\widetilde{m}_{r,r} - m_{r,r}) \dfrac{1}{(n+t-r)!}$ should be zero, $\forall\ t > \max\{s, u, 1\}$; which is not possible. $\square$

Remark 5. From the above recursion equations we notice that for an arbitrary η, just one change in d, say, at k th stage leads to change in each subsequent M_k.

4. Shift-operators and $S_{p,d}$

In this section we discuss how differential operator representation of $S_{p,d}$ helps in concluding facts about shift operators. Shift operator, $\tau_{a,b}$ for $(a,b) \neq (1,0)$, is an infinite order differential operator η with $M_k(x) = [(a-1)x+b]^k/k!$, $k \in \mathbb{N}_0$. We give below a characterization of shift operators having an OPS as sequence of eigen vectors via identification with $S_{p,d}$ and observe that it is a very restricted class.

Theorem 1. *Let $p \in \mathcal{P}_o$, $d \in D$, $a, b \in \mathbb{R} : a \neq 0$. Then $S_{p,d} = \tau_{a,b} \iff d_n = (-1)^n$, $n \in \mathbb{N}_0$, $a = -1$, b-arbitrary and $p_n = q_n \circ \tau_{1,b/2}$, $n \in \mathbb{N}_0$, for some symmetric OPS q.*

Proof. $S_{p,d} = \tau_{a,b}$ on $\mathcal{P}_r$ for some $a, b \in \mathbb{R}$, $a \neq 0$

$\iff S_{p,d}(p_n) = \tau_{a,b}(p_n)$, $n \in \mathbb{N}_0$

$\iff d_n p_{nn} = p_{nn} a^n$, $n \in \mathbb{N}_0$ and other corresponding equalities hold

$\iff d_n = a^n$, $n \in \mathbb{N}_0$ and other corresponding equalities hold.

Next, taking $S = S_{p,d}$, $s = 1$ in Proposition MSZ, Section 2 we have

$$S_{p,d} = \tau_{a,b}$$

$\implies$ if (a_n), (b_n), (c_n) are the recurrence co-efficients for $(p_n)_{n\in\mathbb{N}_0}$, then (a_n/a), $((b_n - b)/a)$, (c_n/a) are the recurrence co-efficients for $(S_{p,d}(p_n))$

$$\implies a_n \left(d_n - \frac{d_{n+1}}{a}\right) p_{n+1}(x) + d_n \left(b_n - \frac{b_n}{a} + \frac{b}{a}\right) p_n(x)$$
$$+ c_n \left(d_n - \frac{d_{n-1}}{a}\right) p_{n-1}(x) = 0,\ n \in \mathbb{N}_0,\ x \in \mathbb{R}$$

$\implies$ for $n \in \mathbb{N}_0$, $d_n - d_{n+1}/a = 0$ as well as $b_n(1 - 1/a) + b/a = 0$ and for $n \in \mathbb{N}$, $d_n - d_{n-1}/a = 0$.

$\implies d_n = a^n$ and for $a \neq 1, b_n = \dfrac{-b}{(a-1)}$, for $n \in \mathbb{N}_0$. For $n \in \mathbb{N}$, $d_n = a^{-n}$.

$\implies a = -1$, $d_n = (-1)^n$, $b_n = b/2$, $n \in \mathbb{N}_0$.

Thus $S_{p,d} = \tau_{a,b}$ implies $d_n = (-1)^n$, $n \in \mathbb{N}_0$, $a = -1$ and
$xp_n(x) = a_n p_{n+1}(x) + \dfrac{b}{2} p_n(x) + c_n p_{n-1}(x)$, $n \in \mathbb{N}_0$, $x \in \mathbb{R}$.

Now,

$xp_n(x) = a_n p_{n+1}(x) + b/2\, p_n(x) + c_n p_{n-1}(x)$, $n \in \mathbb{N}_0$, $x \in \mathbb{R}$

$\implies (x - b/2)p_n(x) = a_n p_{n+1}(x) + c_n p_{n-1}(x)$, $n \in \mathbb{N}_0$, $x \in \mathbb{R}$

$\implies tp_n(t+b/2)= a_n p_{n+1}(t+b/2) + c_n p_{n-1}(t+b/2),\ n \in \mathbb{N}_0,\ t \in \mathbb{R}.$

$\implies q_n = p_n \circ \tau_{1,b/2}$ is symmetric.

It is easy to see that the conditions are sufficient also. $\square$

5. Spectral properties of the differential operator $\widehat{\eta}$ and some matrix operators identified with $E_{p,d}$

Having identified all differential operators $\widehat{\eta}$ in Section 3 we investigate spectral properties of such operators along with those of some matrix operators. All known finite-order differential operators having an OPS as a sequence of eigen functions corresponding to a sequence of non-zero eigen values are examples of $\widehat{\eta}$. Throughout we treat $\widehat{\eta}$ as an operator in some $H(Q)$.

Discussion 2. (i) First, we note that one can obtain the adjoint $\widehat{\eta}^*$ for an arbitrary $\widehat{\eta} \approx E_{p,d},\ p \in \mathcal{P},\ d \in \widetilde{D}$ completely in terms of the coordinates of an element in $H(Q)$, Q a PS, an OPS or an ONS and the sequence d. Also, closure may be obtained in similar terms in some cases since, for a linear operator S in $H(Q)$ having $Q_n,\ n \in \mathbb{N}_0$ in the domain of S,

$$g \in D(S^*) \iff \sum_{k=0}^{\infty} |\langle g, S(Q_k)\rangle|^2, \text{ is finite.}$$

For $g \in D(S^*),\ S^*g = \sum\limits_{k=0}^{\infty} \langle g, S(Q_k)\rangle Q_k.$

(ii) As an application, we consider the following four classes of operators and observe that each case presents a radically different situation.
- (a) $E_{L_n^\alpha,d}$ in $H(\widetilde{L}_n^{\alpha+1}),\ \alpha > 0$
- (b) $E_{L_n^{\alpha+1},d}$ in $H(\widetilde{L}_n^{\alpha}),\ \alpha > -1$
- (c) $E_{L_n^\alpha,d}$ in $H(L_n^{\alpha+1}),\ \alpha > -1$
- (d) $E_{L_n^{\alpha+1},d}$ in $H(L_n^{\alpha}),\ \alpha > -1.$

(iii) For notational convenience, from here onwards in this section, we refer to any one of the above operators as T. We will see in the theorems that follow that all the above operators are closable with appropriate conditions on d; hence, $D(T^*)$ is dense. Yet, we observe that elements common to the $D(T^*)$ and the space of polynomials may vary anywhere from zero to the full space. In case (a) all basic vectors are always present in the domain of T^* independent of what d is. Hence, in this case, we have the closure for all operators $E_{L_n^\alpha,d}$ in$H(\widetilde{L}_n^{\alpha+1})$, in particular, for $m^\alpha,\ \alpha > 0$ in $H(\widetilde{L}_n^{\alpha+1})$. In contrast, in (b) either all basic vectors are present or none is in the domain of T^*. For matrix

operator in (c), the situation is entirely different, only a few basic vectors may be in $D(T^*)$ or may be none. For operator in (d) the situation is same as that in (b).

(iv) It may be emphasized that the same analysis may be applied to many more differential operators satisfying the basic requirement of explicit availability of connection co-efficients.

Theorem 2. *Let T be $E_{L_n^\alpha,d}$ in $H(\widetilde{L}_n^{\alpha+1})$, $d \in \widetilde{D}$, $q = (L_n^{\alpha+1})$, $\alpha > 0$. Let $g \in H(\widetilde{q})$. Set $\ell = \sum_{k=0}^{\infty}(g_k/r_k^{\alpha+1})$ with $r_k^{\alpha+1} = \|L_k^{\alpha+1}\|$, usual norm.*
Then we have the following

(i) $g \in D(T^*) \iff \sum_{k=0}^{\infty} \left| g_k \bar{d}_k + \sum_{t=0}^{k-1} (\bar{d}_t - \bar{d}_{t+1}) \frac{r_t^{\alpha+1}}{r_k^{\alpha+1}} g_t \right|^2$ *is finite*,

(ii) *for* $g \in D(T^*)$, $T^* g = \sum_{k=0}^{\infty} \left(g_k \bar{d}_k + \sum_{t=0}^{k-1} (\bar{d}_t - \bar{d}_{t+1}) \frac{r_t^{\alpha+1}}{r_k^{\alpha+1}} g_t \right) \widetilde{q}_k$,

(iii) $\widetilde{q}_s \in D(T^*)$, $s \in \mathbb{N}_0$,

(iv) T *is closable and*

$$g \in D(\overline{T}) \iff \sum_{s=0}^{\infty} \left| g_s d_s + \sum_{k=s+1}^{\infty} (g_k/r_k^{\alpha+1}) r_s^{\alpha+1} (d_s - d_{s+1}) \right|^2 \text{ is convergent} \iff$$

$$\sum_{s=0}^{\infty} \left| g_s d_s + r_s^{\alpha+1}(d_s - d_{s+1}) \left(\ell - \sum_{k=0}^{s} (g_k/r_k^{\alpha+1}) \right) \right|^2 \text{ is finite},$$

(v) *for* $g \in D(\overline{T})$, $\overline{T} g = \sum_{s=0}^{\infty} \left(g_s d_s + r_s^{\alpha+1}(d_s - d_{s+1}) \left(\ell - \sum_{k=0}^{s} (g_k/r_k^{\alpha+1}) \right) \right) \widetilde{q}_s$,

(vi) $g \in D(\overline{m^\alpha}) \iff \sum_{s=0}^{\infty} \left| g_s(-2s+1) + 2 r_s^{\alpha+1} \left(\ell - \sum_{k=0}^{s} \frac{g_k}{r_k^{\alpha+1}} \right) \right|^2$ *is finite.*

For $g \in D(\overline{m^\alpha})$, $\overline{m^\alpha}(g) = \sum_{s=0}^{\infty} \left(g_s(-2s+1) + 2 r_s^{\alpha+1} \left(\ell - \sum_{k=0}^{s} \left(\frac{g_k}{r_k^{\alpha+1}} \right) \right) \right) \widetilde{q}_s$.

Proof. From Matrix form in Example 1, Appendix C.3 , we have for $k \in \mathbb{N}_0$,

$$T(\widetilde{q}_k) = d_k \widetilde{q}_k + \sum_{t=0}^{(k-1)} (d_t - d_{t+1}) \frac{r_t^{\alpha+1}}{r_k^{\alpha+1}} \widetilde{q}_t.$$

This gives (i) and (ii).
Discussion 2 (i) , then gives

$\widetilde{q}_s \in D(T^*) \Longleftrightarrow \sum\limits_{k=0}^{\infty} \left| (\bar{d}_s - \bar{d}_{s+1}) \dfrac{r_s^{\alpha+1}}{r_k^{\alpha+1}} \right|^2$ is finite.

Now $(1/r_k^{\alpha+1})_{k\in\mathbb{N}_0} \in \ell_2$, refer to Appendix A.3.

Therefore $\widetilde{q}_s \in D(T^*)$, $s \in \mathbb{N}_0$, making $D(T^*)$ dense.

Hence T is closable. Also, $T^*(\widetilde{q}_s) = \bar{d}_s \widetilde{q}_s + \sum\limits_{k=s+1}^{\infty} (\bar{d}_s - \bar{d}_{s+1}) \dfrac{r_s^{\alpha+1}}{r_k^{\alpha+1}} \widetilde{q}_k$.

Since for a closable operator $T, \overline{T} = T^{**}$ (iv) and (v) follow.

Part (vi) follows from Remark 1(iv)(b), since in this case, $d_s - d_{s+1} = 2$, $s \in \mathbb{N}_0$.

Theorem 3. *Let T be $E_{L_n^{\alpha+1},d}$ in $H(\widetilde{q})$, $q = (L_n^{\alpha})_{n\in\mathbb{N}_0}$, $\alpha > -1$, $d \in \widetilde{D}$. Let $g \in H(\widetilde{q})$. Then*

(i) $g \in D(T^*) \Longleftrightarrow \sum\limits_{k=0}^{\infty} \left| \bar{d}_k g_k + \sum\limits_{t=0}^{(k-1)} (\bar{d}_k - \bar{d}_{k-1}) \dfrac{r_t^{\alpha} g_t}{r_k^{\alpha}} \right|^2$ *is finite,*

(ii) *for* $g \in D(T^*), T^*g = \sum\limits_{k=0}^{\infty} \left(\bar{d}_k g_k + \sum\limits_{t=0}^{(k-1)} (\bar{d}_k - \bar{d}_{k-1}) g_t \dfrac{r_t^{\alpha}}{r_k^{\alpha}} \right) \widetilde{q}_k$,

(iii) *for* $s \in \mathbb{N}_0$, $\widetilde{q}_s \in D(T^*) \Longleftrightarrow ((\bar{d}_k - \bar{d}_{k-1})/r_k^{\alpha})_{k\in\mathbb{N}_0} \in \ell_2$,

(iv) *for* $d \in \widetilde{D} : ((\bar{d}_k - \bar{d}_{k-1})/r_k^{\alpha})_{k\in N_0} \in \ell_2$,

  (a) *T is closable,*

  (b) $g \in D(\overline{T}) \Longleftrightarrow \sum\limits_{s=0}^{\infty} \left| g_s d_s + \sum\limits_{k=s+1}^{\infty} (d_k - d_{k-1}) \dfrac{r_s^{\alpha}}{r_k^{\alpha}} g_k \right|^2$ *is finite,*

  (c) *for* $g \in D(\overline{T})$, $\overline{T}g = \sum\limits_{s=0}^{\infty} \left(g_s d_s + \sum\limits_{k=s+1}^{\infty} (d_k - d_{k-1}) \dfrac{r_s^{\alpha}}{r_k^{\alpha}} g_k \right) \widetilde{q}_s$,

(v) *for* $\alpha > 1$,

  (a) $g \in D(\overline{m^{\alpha+1}}) \Longleftrightarrow \sum\limits_{s=0}^{\infty} \left| g_s(-2s+1) + \sum\limits_{k=s+1}^{\infty} (-2) \dfrac{r_s^{\alpha}}{r_k^{\alpha}} g_k \right|^2$ *is finite,*

  (b) *for* $g \in D(\overline{m^{\alpha+1}})$, $\overline{m^{\alpha+1}}(g) = \sum\limits_{s=0}^{\infty} \left(g_s(-2s+1) + \sum\limits_{k=s+1}^{\infty} (-2) \dfrac{r_s^{\alpha}}{r_k^{\alpha}} g_k \right) \widetilde{q}_s$.

Proof. It is enough to note that for $k \in \mathbb{N}_0$,

$$T(\widetilde{q}_k) = d_k \widetilde{q}_k + \sum_{t=0}^{(k-1)} (d_k - d_{k-1}) \frac{r_t^{\alpha}}{r_k^{\alpha}} \widetilde{q}_t \,,$$

and for d as in (iv), T is closable by part (iii). The remaining parts may be proved as earlier.

For part (v) we note that from Remark 1(iv)b, we have $m^{\alpha+1} \equiv T$ in $H(L_n^\alpha)$ for $d = (-2n+1)$. For this d, $d_k - d_{k-1} = \begin{cases} -2 & , \ k \in \mathbb{N} \\ 1 & , \ k = 0\,. \end{cases}$

Therefore $m^{\alpha+1}$ is closable in $H(L_n^\alpha)$ if $\alpha > 1$. $\square$

Theorem 4. *Let $T = E_{p,d}$ be the operator defined via $p = (L_n^\alpha)_{n\in\mathbb{N}_0}$ in $H(q)$, $q = (L_n^{\alpha+1})_{n\in\mathbb{N}_0}$, $\alpha > -1$, $d \in \widetilde{D}$. Let $g \in H(q)$. Then*

(i) $g \in D(T^*) \Longleftrightarrow \sum\limits_{k=0}^{\infty} \left| g_k \bar{d}_k + \sum\limits_{t=0}^{(k-1)} (\bar{d}_t - \bar{d}_{t+1}) g_t \right|^2$ *is finite,*

(ii) *for* $g \in D(T^*)$, $T^* g = \sum\limits_{k=0}^{\infty} \left[g_k \bar{d}_k + \sum\limits_{t=0}^{(k-1)} (\bar{d}_t - \bar{d}_{t+1}) g_t \right] q_k$,

(iii) *for a fixed* $j \in \mathbb{N}_0$, $q_j \in D(T^*) \Longleftrightarrow \bar{d}_j - \bar{d}_{j+1} = 0$,

(iv) $E_{L_n^\alpha, d}$ *is unbounded* $\forall \ d \in \widetilde{D}$ *as an operator in* $H(L_n^{\alpha+1})$.

Proof. From $T(q_k) = d_k q_k + \sum\limits_{t=0}^{(k-1)} (d_t - d_{t+1}) q_t$, we have the first three parts following.

(iv) By definition $d \in \widetilde{D}$ is assumed to be a non-constant sequence. Part (iii) above, therefore, gives that atleast one $q_j, j \in \mathbb{N}_0$, does not belong to $D(T^*)$. Therefore, $D(T^*) \subsetneq H(q)$, $\forall \ d \in \widetilde{D}$. Now, if T is bounded for some $d \in \widetilde{D}$, then T is closable with $D(\overline{T})$ as the whole space $H(q)$ and $\overline{T}$ bounded. This, in turn, implies that $D(T^*) = D(\overline{T}^*)$ as the full space $H(q)$. But this is contradictory. Therefore, T is unbounded for every $d \in \widetilde{D}$. $\square$

Remark 6. Matrix operators in the above theorem are closable for many d's ; for instance, when $d_k - d_{k+1} = d_{k+2} - d_{k+3}, k \in \mathbb{N}_0$, then $E_{p,d}$ is closable being thin. In such cases the spectrum is the whole of complex plane with major part as the continuous spectrum.

Theorem 5. *Let $T = E_{p,d}$, in $H(q)$, $p = (L_n^{\alpha+1})_{n\in\mathbb{N}_0}$, $q = (L_n^\alpha)_{n\in\mathbb{N}_0}$, $\alpha > -1$, $d \in \widetilde{D}$. Let $g \in H(q)$. Then*

(i) $g \in D(T^*) \Longleftrightarrow \sum\limits_{k=0}^{\infty} \left| g_k \bar{d}_k + (\bar{d}_k - \bar{d}_{k-1}) \sum\limits_{t=0}^{(k-1)} g_t \right|^2$ *is finite,*

(ii) *for* $g \in D(T^*)$, $T^* g = \sum\limits_{k=0}^{\infty} \left(g_k \bar{d}_k + (\bar{d}_k - \bar{d}_{k-1}) \sum\limits_{t=0}^{k-1} g_t \right) q_k$,

(iii) *for* $s \in \mathbb{N}_0$, $q_s \in D(T^*) \Longleftrightarrow (\bar{d}_k - \bar{d}_{k-1})_{k\in\mathbb{N}_0}$ *is in* ℓ_2.,

(iv) *for d such that* $(\bar{d}_k - \bar{d}_{k-1})_{k\in\mathbb{N}_0} \in \ell_2$,

(a) $g \in D(\overline{T}) \Longleftrightarrow \sum\limits_{s=0}^{\infty} \left| g_s d_s + \sum\limits_{k=s+1}^{\infty} (d_k - d_{k-1}) g_k \right|^2$ *is finite,*

(b) *for* $g \in D(\overline{T})$, $\overline{T}g = \sum\limits_{s=0}^{\infty} \left(g_s d_s + \sum\limits_{k=s+1}^{\infty} (d_k - d_{k-1}) g_k \right) q_s$.

Proof. Parts (i)-(iv) may be proved as earlier. We may note that by Theorem 3.3 in [7], given as Appendix item D.3 and Matrix form in Example 2, Appendix C.3, T is closable. □

Remark 7. (i) For a closable operator T, $\sigma_0(\overline{T}) = \{\lambda \in \mathbb{C} : E_\lambda^{(2)} \not\equiv o\}$, where $E_\lambda^{(2)}$ is the λ-approximate-Hilbert eigenspace of T as defined in [7] given as Appendix item D.2. Here we note that in case $\overline{T}$ is completely determinable as above, $(\overline{T} - \lambda I)g = 0 \Longleftrightarrow t_s - \lambda g_s = 0, s \in \mathbb{N}_0$, where $\overline{T}g = \sum\limits_{k=0}^{\infty} t_k q_k$. For instance, in Theorem 5, $\lambda \neq d_k$, $k \in \mathbb{N}_0$,

$$(\overline{T} - \lambda I)g = o \Longleftrightarrow g_s = -\sum_{k=s+1}^{\infty} \frac{(d_k - d_{k-1})}{(d_s - \lambda)} g_k, \ s \in \mathbb{N}_0 .$$

(ii) It may be emphasized that every PS or OPS serves as a sequence of eigen functions for an infinite number of formal infinite order differential operators. The adjoint operators for these $\widehat{\eta}$ may be obtained in ℓ_2 or in some weighted $\mathcal{L}^2$ spaces. Closures can, similarly be obtained subject to conditions on d in many cases.

(iii) As has been shown in Theorem 5 that when $d \in \widetilde{D}$ is such that $(\bar{d}_k - \bar{d}_{k-1})_{k \in \mathbb{N}_0}$ belongs to ℓ_2, we have the closure fully determined. In Theorem 6 and Theorem 7,we take $d \in \widetilde{D}$ such that $(d_k - d_{k-1})_{k \in \mathbb{N}_0} \notin \ell_2$. Here we observe heavy dependence of closure of the operator on the sequence of eigen values d.

Theorem 6. ***A set of necessary conditions:*** *Let p, q, T be as in Theorem 5, $d \in \widetilde{D}$. If $(f, g) \in \overline{G(T)}$, then*

(a) $\exists$ $(h_{n,u})_{n,u \in \mathbb{N}_0}$ *such that*
 (a1) $h_{n,u} = 0$, $\forall$ $u > n$,
 (a2) *for* $u \in \mathbb{N}_0$, $\lim\limits_{n \to \infty} h_{n,u} = f_u$,
 (a3) $\lim\limits_{n \to \infty} h_{n,n} d_n = 0$,
 (a4) $\lim_{n \to \infty} \sum\limits_{u=1}^{n} h_{n,u}(d_u - d_{u-1}) = g_0 - f_0 d_0$,

(b) *for* $k \in \mathbb{N}$,

$$g_k = g_0 - f_0 d_0 + f_k d_k - \sum_{u=1}^{k} f_u (d_u - d_{u-1}) .$$

Proof. The statement is a special case of Theorem 3.2 [7], refer Appendix D.5. Proof is as done earlier and is based on (1), (2) and (3) below

(1) For

$$h_n = \sum_{u=0}^{n} h_{n,u} q_u \text{ in the linear span of } \quad q_n\text{'s}$$

$$Th_n = \sum_{k=0}^{n} t_{n,k} q_k,$$

where

$$t_{n,k} = \begin{cases} \sum\limits_{u=k+1}^{n} h_{n,u}(d_u - d_{u-1}) + h_{n,k} d_k & , \quad 0 \le k \le (n-1), \\ h_{n,n} d_n & , \quad k = n\,. \end{cases}$$

(2) $(f,g) \in \overline{G(T)} \Rightarrow \exists\ (h_n)$ in $H(q)$ such that $(h_n) \to f$ and $(Th_n) \to g$ in $H(q)$.

(3) T is closable by the above Theorem. Therefore, g is unique. □

Theorem 7. ***A set of sufficient conditions :*** *Let p, q, T be as in Theorem 5, $d \in \widetilde{D}$. Let $f \in H(q)$ be such that*

(i) $\lim\limits_{n\to\infty} \left[\sum\limits_{u=1}^{n} f_u (d_u - d_{u-1}) \right]$ *is finite and, say, equals S,*

(ii) *for $(g_k)_{k\in\mathbb{N}_0}$ as $g_0 = S + f_0 d_0$,*

$$g_k = S - \sum_{u=1}^{k} f_u (d_u - d_{u-1}) + f_k d_k, \quad k \ge 1,$$

$(g_k)_{k\in\mathbb{N}_0}$ is in ℓ_2,

(iii) $\lim\limits_{n\to\infty} (n+1)|f_n d_n - g_n|^2 = 0$.

Then $f \in D(\overline{T})$ and $g = \lim\limits_{n\to\infty} \left(\sum\limits_{t=0}^{n} g_t q_t \right)$ is the unique element of $H(q)$ such that $(f,g) \in G(\overline{T})$.

Proof. Define $r_{n,u}$, $h_{n,u}$ for $n, u \in \mathbb{N}_0$ as

$$r_{n,u} = \begin{cases} \frac{1}{n^2 2^n (|d_u - d_{u-1}| + |d_u| + 1)} & , \quad 0 \le u \le n \\ -f_u & , \quad u > n \end{cases}$$

$$h_{n,u} = f_u + r_{n,u}.$$

Then

$$h_n = \lim_{k\to\infty} \left(\sum_{u=0}^{k} h_{n,u} q_u \right) = \sum_{u=0}^{n} h_{n,u} q_u$$

is in $\mathcal{P}_c$, $\forall\ n \in \mathbb{N}_0$. Also, $\lim\limits_{n\to\infty} \|h_n - f\|^2 = 0$.

Let $n \in \mathbb{N}_0$. Express $T(h_n)$ as in part (1) in the proof of Theorem 6. Now,

$$\begin{aligned}
\sum_{k=0}^{n} |t_{n,k} - g_k|^2 \le & \left|\sum_{u=1}^{n} r_{n,u}(d_u - d_{u-1})\right|^2 + \left|\sum_{u=n+1}^{\infty} f_u(d_u - d_{u-1})\right|^2 + |r_{n,0}d_0|^2 \\
& + \sum_{k=1}^{(n-1)} \left|\sum_{u=k+1}^{n} r_{n,u}(d_u - d_{u-1})\right|^2 + \sum_{k=1}^{(n-1)} \left|\sum_{u=n+1}^{\infty} f_u(d_u - d_{u-1})\right|^2 \\
& + \sum_{k=1}^{(n-1)} |r_{n,k}d_k|^2 + |r_{n,n}d_n|^2 + \left|\sum_{u=n+1}^{\infty} f_u(d_u - d_{u-1})\right|^2 \\
\le & (n+1)\left|\sum_{k=n+1}^{\infty} f_k(d_k - d_{k-1})\right|^2 \\
& + \text{remainder terms from definition of } r_{n,k}.
\end{aligned}$$

The remainder terms tend to zero as n tends to infinity by the definition of $r_{n,k}$. Hence, from the assumptions, we have

$$\begin{aligned}
0 &\le \limsup_{n\to\infty} \sum_{k=0}^{n} |t_{n,k} - g_k|^2 \\
&\le \limsup_{n\to\infty} (n+1)\left|\sum_{k=n+1}^{\infty} f_k(d_k - d_{k-1})\right|^2 \\
&= \lim_{n\to\infty} (n+1)|f_n d_n - g_n|^2 \\
&= 0
\end{aligned}$$

$\Rightarrow \lim_{n\to\infty} \|Th_n - g\|^2 = 0$. Hence, $\exists\ (h_n)_{n\in\mathbb{N}_0} \in \mathcal{P}_c : (h_n) \to f,\ (Th_n) \to g$ implying $(f, g) \in \overline{G(T)}$. This $g \in H(q)$ is unique, since T is closable. $\square$

Acknowledgments. I thank Professor Ajit Iqbal Singh for continued support and insightful suggestions.

Appendix A.
Notation and terminology

- Let $\mathbb{N}_0$ be the set of non-negative integers and $\mathbb{N}$, the set of natural numbers. Let $\mathbb{R}$ be the set of Real numbers and $\mathbb{C}$ be the set of complex numbers.
- $p(n, r) = \begin{cases} \frac{n!}{(n-r)!}, & n \neq 0 \\ 0 & n = 0 \end{cases}.$
- A function is usually denoted by f, g, y etc. but at times, in keeping with notation in a source, we write them as $f(x)$, $g(x)$, $y(x)$ etc. The context makes it clear.

- For $n \in \mathbb{N}_0$, let (e_n) be the sequence $(\delta_{nm})_{m\in\mathbb{N}_0}$.
- Let w be the space of all complex sequences and φ be the subspace of sequences with only finitely many non-zero terms. We equip ω with the topology of pointwise convergence which is also given by the translation invariant metric
$$d(x,y) = \sum_{n\in\mathbb{N}_0} \frac{|x_n - y_n|}{2^n(1+|x_n-y_n|)}.$$
Then (ω, d) is complete.
- Let l_2 be the subspace of ω consisting of square-summable sequences which becomes a Hilbert space under the usual inner product $\langle x,y\rangle = \sum_{n\in\mathbb{N}_0} x_n\bar{y}_n$.
- We denote by $\widetilde{D}$ the set of sequences $d = (d_n)_{n\in\mathbb{N}_0}$ of non-zero complex scalars, $d \in \widetilde{D}$ is assumed to be non-constant to avoid triviality. Let $D = \{\ d \in \widetilde{D} : d$ is a real sequence with $d_0 = 1\}$. Set $d_{-1} = 0$. For $d \in \widetilde{D}$, closure of the set $\{d_n : n \in \mathbb{N}_0\}$ is denoted by $cl(d)$.

A.1. **Some specific orthogonal polynomial sequences.**

(i) For $\alpha > -1$, $(L_n^{(\alpha)})_{n\in\mathbb{N}_0}$: the Generalised Laguerre polynomial sequence is given by $L_n^{(\alpha)}(x) = \sum_{k=0}^n \dfrac{(-1)^k}{k!}\binom{n+\alpha}{n-k}x^k$, where $\binom{t}{0} = 1$ and $\binom{t}{k} = \dfrac{t(t-1)\dots(t-k+1)}{k!}$ for $t, k \in \mathbb{N}$.

(ii) For $-1 < \alpha < \infty, -1 < \beta < \infty$, $(P_n^{\alpha,\beta})_{n=0}^\infty$: the Jacobi Polynomial Sequence is given by $P_n^{\alpha,\beta}(x) = 1/2^n \sum_{m=0}^n \binom{n+\alpha}{m}\binom{n+\beta}{n-m}(x-1)^{n-m}(x+1)^m$.

(iii) $(T_n)_{n\in\mathbb{N}_0}$: the Tchebyshev polynomials of the first kind are given by
$$T_n(x) = \cos(n\theta), \quad x = \cos\theta, \quad 0 \le \theta \le \pi.$$

(iv) $(U_n)_{n\in\mathbb{N}_0}$: the Tchebyshev polynomials of the second kind are given by $U_n(x) = \dfrac{\sin(n+1)\theta}{\sin\theta}$, $x = \cos\theta$, $0 < \theta < \pi$ extended by continuity to $[-1,1]$ so that
$$U_n(-1) = (-1)^n(n+1) \quad \text{and} \quad U_n(1) = n+1.$$

A.2. **Connecting relations.** Given $p = (p_n)_{n\in\mathbb{N}_0}$, a PS, we set $p_{-1} = 0$, for notational convenience.

(i) $L_n^{(\alpha)} = L_n^{(\alpha+1)} - L_{n-1}^{(\alpha)}$, $n \in \mathbb{N}_0$.

(ii) $L_n^{(\alpha+1)} = \sum_{j=0}^n L_j^{(\alpha)}$, $n \in \mathbb{N}_0$.

(iii) $2T_n = U_n - U_{n-2}$, $n \in \mathbb{N}$.
$T_0 = U_0$.

(iv) $U_{2n} = T_0 + 2\sum_{k=1}^n T_{2k}$, $n \in \mathbb{N}$,
$U_0 = T_0$.

$U_{2n+1} = 2\sum_{k=0}^{n} T_{2k+1}$, $n \in \mathbb{N}_0$.

A.3. **An application of Raabe's test to Laguerre polynomials.** For Laguerre polynomials L_k^{β}, $\beta > -1$, $k \in \mathbb{N}_0$, we have the usual norm, $r_k^{\beta} = \|L_k^{\beta}\| = \sqrt{\dfrac{\Gamma(k+\beta+1)}{p(k,k)\Gamma(\beta+1)}}$. Thus $r_k^{\beta} = \begin{cases} \sqrt{(1+\beta)(1+\beta/2),\ldots,(1+\beta/k)} & , \ k \in \mathbb{N} \\ 1 & , \ k = 0\,. \end{cases}$

If $\beta > 1$, then by Raabe's Test $\left(\frac{1}{r_k^{\beta}}\right) \in \ell_2$.

Appendix B. Formal Differential operator

B.1. η **; Formal differential operator.**

(i) Let $\eta : \mathcal{P}_c \longrightarrow \mathcal{P}_c$ be the operator defined by the formal sum

$$\eta(y)(x) = \sum_{k=0}^{\infty} M_k(x) y^{(k)}(x)$$

where $M_k(x) \in \mathcal{P}_c$ is such that $\deg M_k(x) \leq k$ for $k \in \mathbb{N}$, $M_0(x)$ is a constant, $y^{(0)} = y$, $y^{(k)}$ is the k^{th} derivative of $y \in \mathcal{P}_c$ for $k \geq 1$. For $p \in \mathcal{P}$, $d \in \widetilde{D}$ equations $\eta p_n = d_n p_n$, $n \in \mathbb{N}_0$ collectively are abbreviated as $\eta p = dp$.

For $n \in \mathbb{N}_0$, when p_n is taken as a solution of $\eta p_n = d_n p_n$, $n \in \mathbb{N}_0$, it will be assumed that deg p_n=n. Also d_n will be referred to as an eigen-value and p_n as the corresponding eigen-function.

(ii) m ; Let $m : \mathcal{P}_c \longrightarrow \mathcal{P}_c$ be the operator η for which $M_k(x) \equiv 0$, $\forall\ k > r$ for some smallest $r \in \mathbb{N}$ with $M_r \not\equiv 0$. Number r is then referred to as the order of finite order differential operator m.

(iii) $\widehat{\eta}, \widehat{m}$; $\widehat{\eta}$ is an operator η having a PS as a sequence of eigen functions corresponding to a sequence of non-zero eigenvalues. Similar meaning is attached to $\widehat{m}$.

Study of arbitrary m is implicit while studying η in general. Similarly treatment of $\widehat{m}$ automatically comes under any analysis done for $\widehat{\eta}$.

B.2. **An infinite order differential operator.** We give below details of a well-known infinite order differential operator η. We refer to [4], Chapter XVII for more information on the topic.

For $\alpha > -1, K > 0$, the Generalized Laguerre-type polynomials $(L_n^{\alpha,K})_{n=0}^{\infty}$ are defined thus,

$$L_n^{\alpha,K}(x) = \left[1 + K\binom{n+\alpha}{n-1}\right] L_n^{\alpha}(x) + K\binom{n+\alpha}{n}\frac{d}{dx}L_n^{\alpha}(x), n \in \mathbb{N}_0,$$

where $L_n^\alpha(x), n \in \mathbb{N}_0$ are the Generalized Laguerre polynomials. Let

$$\eta_{\alpha,K}(y)(x) = \sum_{k=0}^{\infty} M_k(x) y^{(k)}(x)$$

, where

$$M_k(x) = \begin{cases} K\frac{1}{k!}\sum\limits_{j=1}^{k}(-1)^{k+j+1}\binom{\alpha+1}{j-1}\binom{\alpha+2}{k-j}(\alpha+3)_{k-j}x^k & , \quad k \geq 2, \\ -Kx+\alpha+1 & , \quad k=1, \\ 1 & , k=0. \end{cases}$$

Take α, K such that $d_n = -K\binom{n+\alpha+1}{n-1} - n + 1 \neq 0, n \in \mathbb{N}_0$. Let $p_n = L_n^{\alpha,K}, n \in \mathbb{N}_0$. Then $\eta_{\alpha,K}(p_n) = d_n p_n, n \in \mathbb{N}_0$.

B.3. **Differential Operators associated with some classical OPS.**

(i) Let $-1 < \alpha < \infty, -1 < \beta < \infty, \alpha + \beta \neq -1, p = (P_n^{\alpha,\beta})$. Let $m^{\alpha,\beta}$ be the differential operator determined by

$$(m^{\alpha,\beta}f)(x) = (1-x^2)f''(x) + (\beta - \alpha - (\alpha+\beta+2)x)f'(x) + f(x).$$

For

$$d_n = -n(n+\alpha+\beta+1)+1, n \in \mathbb{N}_0$$
$$m^{\alpha,\beta}(p_n) = d_n p_n, n \in \mathbb{N}_0.$$

Condition $\alpha + \beta \neq -1$ is imposed to have $d_n \neq 0, n \in \mathbb{N}_0$.

(ii) Let $p = (H_n)_{n=0}^{\infty}$, the Hermite polynomial system, $d = (-2n+1)_{n\in\mathbb{N}_0}$ and m^h be the differential operator determined by

$$(m^h f)(x) = f''(x) - 2xf'(x) + f(x).$$

Then $m^h(p_n) = d_n p_n, n \in \mathbb{N}_0$.

(iii) Let $\alpha > -1$ and $p = (L_n^\alpha)_{n=0}^{\infty}$, the Generalised Laguerre system. Let m^α be the differential operator determined by

$$(m^\alpha f)(x) = 2xf''(x) + 2(\alpha+1-x)f'(x) + f(x).$$

Then for $d = (-2n+1)_{n\in\mathbb{N}_0}$,

$$m^\alpha(p_n) = d_n p_n, n \in \mathbb{N}_0.$$

Appendix C.
Hilbert Space $H(Q)$ and Dilation Operators

C.1. **(p,d) dilation operators.**

(i) $S_{p,d}$: For $d \in D$ and $p \in \mathcal{P}$ let $S_{p,d}$ be the linear operator on $\mathcal{P}_c$ to itself given by $S_{p,d}(p_n) = d_n p_n$, $n \in \mathbb{N}_0$. For a fixed p, $S_{p,d}$ and d determine each other. On the other hand, for a fixed d, p is not always determined by the operator $S_{p,d}$ as it may be equal to $S_{q,d}$ for

an infinite number of $q's$. One such d is $d_n = (-1)^n, n \in \mathbb{N}_0$. For this d, $S_{p,d}{=}S_{q,d}$, on $\mathcal{P}_c$ for all $\gamma > -1, p = (P_n^{\gamma,\gamma}), q = (T_n)$.

(ii) Dilation map $E_{p,d}$: When we relax the condition either on p or on d in the definition of $S_{p,d}$ above to having p as an arbitrary $p \in \mathcal{P}$, $d \in \widetilde{D}$, we denote the resulting operator by $E_{p,d}$

C.2. **Hilbert Spaces** $H(q)$**.** Let $q = (q_n)$ be a PS. We consider $\mathcal{P}_c$ as the linear span of $\{e_n = q_n : n \in \mathbb{N}_0\}$ and identify it with φ making $a = (a_n)_{n\in\mathbb{N}_0} \in \varphi$ correspond to $\sum_{n\in\mathbb{N}_0} a_n q_n$. Then the completion of $\mathcal{P}_c$ with the inner product induced by the usual inner product in $\varphi \subset l_2$ is a Hilbert space, which we denote by $H(q)$.

(i) If q is taken as an OPS, then $\tilde{q}$ is the corresponding ONS. When referring to q a PS, an OPS or an ONS, we shall write $\mathbf{Q}$.
An arbitrary $g \in H(Q)$ will be identified with the sequence $(g_j)_{j\in\mathbb{N}_0} \subset l_2$ such that $g = \sum_{j\in\mathbb{N}_0} g_j Q_j$, where $Q = (Q_n)$. Elements of $Q = (Q_n)$ in $H(Q)$ will be referred to as basic vectors.

(ii) When q is an OPS orthogonal with respect to a weight function w with the interval of orthogonality as (a, b) and satisfying the normalizing condition $\int_a^b w(x)dx = 1$, then $H(Q)$ is $\mathcal{L}_\omega^{(2)}(a, b)$. The interval of orthogonality may be infinite . For instance, for $\alpha > -1$, $H(\widetilde{L}_n^\alpha)$ is $\mathcal{L}_\omega^{(2)}(0, \infty)$ with weight $w(x) = \frac{x^\alpha \exp(-x)}{\Gamma(\alpha)}$, for $x > 0$.

(iii) We refer to [1] for orthogonal polynomials.

C.3. **Some matrix Operators.** The linear operator T in $H(q)$ determined by $S_{p,d}$ on $\mathcal{P}$ is given by the matrix $A = (a_{jk})$ in each of the following examples, when p and q are taken as specified.

Example 1. *For $p = (p_n) = (L_n^\alpha)$ and $q = (q_n) = (L_n^{\alpha+1})$, $\alpha > -1$, we have $A = (a_{jk})$ with*

$$
\begin{aligned}
a_{jk} &= 0, \quad k < j,\\
a_{jj} &= d_j, \quad j \geq 0,\\
a_{jk} &= d_j - d_{j+1}, \quad k > j\ .
\end{aligned}
$$

Example 2. *For $p = (p_n) = (L_n^{\alpha+1})$ and $q = (q_n) = (L_n^\alpha)$, $\alpha > -1$, we have $A = (a_{jk})$ with*

$$
\begin{aligned}
a_{jk} &= 0, \quad k < j,\\
a_{jj} &= d_j, \quad j \geq 0,\\
a_{jk} &= d_k - d_{k-1}, \quad 0 \leq j < k,\ k \geq 1.
\end{aligned}
$$

Appendix D.
Thin Matrices and Closability of Associated Operators in Hilbert Spaces

We give below relevant definitions, theorems and examples from the paper [7] by the author and Ajit Iqbal Singh. We shall confine our attention to the setting of Hilbert Space l_2 though the paper begins with that of Locally Convex Space ω before coming to the Hilbert Space setting. An infinite complex matrix $A = (a_{jk})_{j,k\in\mathbb{N}_0}$ determines an operator T from φ to ω in a natural way. Also, it is well-known that Columns of A are in $l_2 \iff T$ takes φ into l_2 .

D.1. **Definition of Thin infinite matrices.**

(i) We define an equivalence relation $\sim$ in ω as : For a, b in ω, $a \sim b$ if and only if $a - \mu b \in l_2$ for some non-zero μ in $\mathbb{C}$.

The relation $\sim$ divides ω into mutually disjoint equivalence classes $[a]$ with $[o] = l_2$. Further for $a \not\sim o$, $a \sim b$, the constant μ is unique.

(ii) For an infinite matrix A denote by a_k, the kth row, $k \in \mathbb{N}_0$. Then there exists an index set $I \subseteq \mathbb{N}_0$ and a decomposition $(N_i)_{i\in I}$ of $\mathbb{N}_0$ into mutually disjoint sets such that for $k, t \in \mathbb{N}_0$, $a_k \sim a_t$ if and only if $k, t \in N_i$ for some $i \in I$.

For notational convenience, if $a_k \in l_2$ for some $k \in \mathbb{N}_0$ then we take $\mathrm{N}_0 = \{t : a_k \sim a_t\}$ and otherwise we take $I \subset \mathbb{N}$. For $i \in I$, let $k_i^0 = \min\{k : k \in N_i\}$ and for $k \in N_i$ with $i \geq 1$, let m_k be the unique non-zero number such that $a_k - m_k a_{k_i^0} \in l_2$.

Clearly, $m_{k_i^0} = 1$.

If $0 \in I$, then we set $m_t = 0$ for t in N_0. Let $m = (m_k)_{k\in\mathbb{N}_0}$.

If for an $i \in I$, $i \geq 1$, we have N_i as an infinite set, then we arrange N_i as a sequence, say, $(t_i)_{t\in\mathbb{N}_0}$ with $0_i = k_i^0$ and $t_i \neq t_i'$ for $t \neq t'$ in $\mathbb{N}_0$.

Next we put $b_{jk} = a_{jk} - m_j a_{k_i^0 k}$ for $j \in N_i$, $i \in I$, $k \in \mathbb{N}_0$. We say that A is $(m, (k_i^0)_{i\in I})$-*thin* with *thinning* $B = (b_{jk}), j, k \in \mathbb{N}_0$.

(iii) (a) Clearly, $V = B$ restricted to l_2 is a continuous linear operator on l_2 into ω.

(b) If I contains an $i \geq 1$, i.e., not all the rows of A are in l_2, then T is not continuous as an operator from l_2 to ω. To see this well-known fact, we first note that $s_n = \sum_{k=0}^{n} |a_{k_i^0 k}|^2 \uparrow \infty$.

Let n_0 be the least integer with $s_{n_0} \neq 0$. We next define the sequence $(h^{(n)})$ in φ by taking $h_t^{(n)} = \dfrac{\overline{a_{k_i^0 t}}}{s_n}$, $n \geq n_0$, $0 \leq t \leq n$, and $h_t^{(n)} = 0$, otherwise. Then $h^{(n)} \to o$ in l_2 whereas, $(Th^{(n)})_{k_i^0} = 1$ for $n \geq n_0$.

(iv) A will be called *thin* if either $I = \{0\}$ or for each $i \in I$, $i \geq 1$, N_i is infinite and the sequence $m^{(i)} = (m_{t_i})_{t\in\mathbb{N}_0} \notin l_2$.

(v) A will be called *blocked* if $a_{jk} = 0$ for j and k not in the same N_i for any $i \in I$.

D.2. **λ-approximate-Hilbert eigenspace.** Suppose columns of A are in l_2. Then, for $\lambda \in \mathbb{C}$, $E_\lambda^{(2)} = \{x : (x, \lambda x) \in \overline{G(T)}\}$ will be called the *λ-approximate-Hilbert eigenspace* of T.

D.3. **Criterion for closability.** Theorem 3.3 from [7]: Let A be an infinite matrix whose columns are in l_2. Then the following hold.

(i) If A is thin then T is closable.
(ii) If A is blocked and T is closable then A is thin.

D.4. **Examples of Thin infinite matrix, infinite blocked matrix.**

(1) Example.2 in Appendix C.3 is thin.
(2) Matrix in this example is blocked.

Let $p = (p_n)$:

$$p_0 = T_0, \quad p_n = 2T_n \qquad \text{for } n \in \mathbb{N}_0,\ n \geq 1$$

and $q = (q_n) = (U_n)$.

Then the matrix for $T = S_{p,d}$ in $H(q)$ is given by $A = (a_{jk})$

$$\begin{aligned} a_{jk} &= 0, \qquad k < j \\ a_{jj} &= d_j, \quad j \geq 0 \\ a_{jk} &= d_j - d_{j+2}, \quad k = j + 2s,\ s \in \mathbb{N} \\ a_{jk} &= 0, \qquad k = j + (2s-1), \quad s \in \mathbb{N}. \end{aligned}$$

Therefore, $T = S_{p,d}$ can be thought of as $T' \oplus T''$ acting on $\mathcal{P}' \oplus \mathcal{P}''$ with

$$\mathcal{P}' = \text{the linear span of } \{q_{2n} : n \in \mathbb{N}_0\} \text{ and}$$
$$\mathcal{P}'' = \text{the linear span of } \{q_{2n+1} : n \in \mathbb{N}_0\},$$

T' and T'' are given by matrices A' and A'' respectively of the same form as A in Example 1 in Appendix C.3, with d replaced by $d' = (d_{2n})_{n\in\mathbb{N}_0}$ and $d'' = (d_{2n+1})_{n\in\mathbb{N}_0}$.

Thus cardinality of $I \leq 3$, where I is an indexing set as in D.1(ii) above. Hence Matrix A is blocked.

D.5. **Theorem 3.2 [7].** Let A be an infinite matrix whose columns are in l_2. Let m, $(k_i^0)_{i\in I}$ and V be as in D.1.(ii) and (iii) above.

(i) Let $(x, y) \in \overline{G(T)}$. Then
$$y_t = y_{k_i^0} m_t + (Vx)_t, \quad t \in N_i,\ i \in I\,.$$
(ii) If $o \neq x \in E_0^{(2)}$ then 0 is an eigenvalue of V with x as an eigenvector.
(iii) For $\lambda \neq 0$ in $\mathbb{C}$ and $o \neq x \in E_\lambda^{(2)}$, $x_{k_i^0} = 0$ for all $i \in I$, $i \geq 1$ if and only if λ is an eigenvalue of V with x as an eigenvector.

(iv) If $o \neq x \in E_\lambda^{(2)}$ then,

$$(\lambda I - V)x = \lambda\big((x_{k_i^0} m_t)_{t \in N_i}\big)_{i \in I}$$

and, thus, for $\lambda \neq 0$ in $\mathbb{C}$ we have $\big((x_{k_i^0} m_t)_{t \in N_i}\big)_{i \in I} \in R(\lambda I - V)$.

(v) If 0 is not an eigenvalue of V then T^{-1} exists and is closable.

(vi) If $\lambda(\neq 0)$ in $\mathbb{C}$ is not an eigenvalue of V with an eigenvector in the subspace $\{x : x_{k_i^0} = 0, i \in I, i \geq 1\}$ and for no non-zero sequence $(\alpha_{k_i^0})_{i \in I}$, $((\alpha_{k_i^0} m_t)_{t \in N_i})_{i \in I}$ is in $R(\lambda I - V)$ then $(\lambda I - T)^{-1}$ exists and is closable.

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